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The method of Darboux transformation matrix for solving the Landau–Lifschitz equation for a spin chain with an easy plane

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Abstract. The Landau–Lifschitz equation for a spin chain with an easy plane is solved by the method of the Darboux transformation matrix. In terms of a particular parameter k, Jost solutions and Darboux matrices are generated in a recursive manner. The Jost solutions are shown to satisfy the corresponding Lax equations by a suitable choice of the constants involved in the Darboux matrices. A system of linear equations is derived and can yield the expressions for multi-soliton solutions. Asymptotic behaviour in the limits as $t \to \pm\infty$ is derived. An expression of the one-soliton solution is given in terms of elementary functions of x and t, as an example.

1. Introduction

The Landau–Lifschitz equation for a spin chain with an easy plane has attracted the attention of many authors in recent decades. However, some difficulties exist. Firstly, the Landau–Lifschitz equation differs from the equation for an isotropic chain; it could not be solved by separating variables in moving coordinates (Tjio and Wright 1977, Quispel and Capel 1983). Secondly, when one tries to solve it by using the inverse transform, in addition to complexity due to the Riemann surface and inquired appearance of a double-valued function of the standard spectral parameter, the reflection coefficient at the edges of cuts in the complex plane could not be neglected even in the reflectionless case, as we shall show.

Mikeska (1978) reduces the equation with an easy plane to the sine–Gordon equation and finds a solution. However, when the applied field tends to zero, the solution becomes a travelling-wave solution which does not obviously relate to the nonlinearity of the spin chain. The solution given by Long and Bishop (1980) and the solution found by Nakamura and Sasada (1982) by means of variation methods do not satisfy the equation by direct substitution.

Kosevich *et al* (1977) reduce the Landau–Lifschitz equation for a spin chain with an easy plane to an approximate equation and then finds a solution. However, this solution cannot be considered as an approximate solution of the Landau–Lifschitz equation for a spin chain with an easy plane, since it does not satisfy the equation in the approximation of the first order of anisotropy.

Bolovik (1978) and Bolovik and Kulinich (1984) tried to derive the Marchenko equation, but their derivation is, as we shall explain later, questionable, and they could not find a one-soliton solution from it.

Bogdan and Kovalev (1980) attempt to construct exact multi-soliton solutions with the direct method of Hirota. However, they could not prove a series of non-trivial identities on the parameters of the solution to the end, and hence explicit expressions for the solutions were not obtained.

For the spin with complete anisotropy (say, $J_3 > J_2 > J_1$), Sklyanin (1979) finds an expression for its Lax pair, and Mikhailov (1982) and Rodin (1983) are able to reduce the problem to the Riemann boundary value problem on a torus and study it by means of the inverse tranform. However, the derivation and the results are expressed in terms of elliptic functions and are more complicated. Even though the soliton solutions were found, they are difficult to transform to those for the case of an easy plane since, say, $J_3 = J_2$ (see Faddeev and Takhtajan 1987). Therefore, an exact treatment of the Landau–Lifschitz equation for a spin chain with an easy plane has never appeared.

In this paper, the equation is solved by means of the method of Darboux transformation matrix. Introducing a particular parameter k, and constructing the Darboux matrix in a recursive manner, we show that the Jost solutions can be generated and then soliton solutions can be obtained in a recursive manner. We give an explicit expression of the one-soliton solution in terms of elementary functions of x and t, as an example. A system of linear algebraic equations which can give expressions for multi-soliton solutions is then derived.

To justify the present method, a particular procedure based upon the well known Liouville theorem is developed to show that the Jost solutions generated in a recursive manner indeed satisfy the corresponding Lax equations.

By using a unitary transformation in spin-space, the system of equations is transformed into a form which is more suitable for determining the expressions of multi-soliton solutions. Asymptotic behaviour of multi-soliton solutions in the limits as $t \to \pm\infty$ is found directly from this system by means of a special procedure (Huang and Chen 1990).

Finally, the results are expressed in terms of a particular parameter ζ, which is convenient for discussing the behaviour of the expressions in the limit as the anisotropy approaches zero. One can see that these expressions tend to those for the isotropic spin chain as the anisotropy vanishes.

In the concluding remarks we discuss reasons why the previous works are unsatisfactory.

2. The Landau–Lifschitz equation with an easy plane

The Landau–Lifschitz equation for a spin chain with an easy plane is

$$S_t = S \times S_{xx} + S \times JS \qquad |S| = 1 \tag{1}$$

where the diagonal matrix J,

$$J = \mathrm{diag}(0, 0, -16\rho^2) \tag{2}$$

characterizes the easy plane, that is the 12-plane. Here ρ is a positive constant and 16 is introduced for later convenience. The Lax pair for this equation is given by

$$L = -\mathrm{i}\mu S_3\sigma_3 - \mathrm{i}\kappa(S_1\sigma_1 + S_2\sigma_2) \tag{3}$$

$$\begin{aligned} M = {} & \mathrm{i}2\kappa^2 S_3\sigma_3 + \mathrm{i}2\kappa\mu(S_1\sigma_1 + S_2\sigma_2) - \mathrm{i}\kappa(S_2S_{3x} - S_3S_{2x})\sigma_1 - \mathrm{i}\kappa(S_3S_{1x} - S_1S_{3x})\sigma_2 \\ & - \mathrm{i}\mu(S_1S_{2x} - S_2S_{1x})\sigma_3 \end{aligned} \tag{4}$$

where parameters μ and κ satisfy

$$\mu^2 = \kappa^2 + 4\rho^2. \tag{5}$$

If one of them is taken as an independent parameter, the other is a double-valued function of it. As we shall see, in development of the Darboux transformation method it is reasonable to introduce an auxiliary parameter k such that

$$\mu = 2\rho\frac{k+k^{-1}}{k-k^{-1}} \tag{6}$$

$$\kappa = 2\rho\frac{2}{k-k^{-1}}. \tag{7}$$

The Lax equations are

$$\partial_x F(k) = L(k)F(k) \tag{8}$$

$$\partial_t F(k) = M(k)F(k). \tag{9}$$

From now on we shall drop the arguments x and t unless necessary.

Since the 12-plane is the easy plane, the asymptotic spin must lie on it and we thus see that

$$S = S_0 = (1,0,0) \tag{10}$$

is the simplest solution of (1). The corresponding Jost solution of (8) and (9) may be chosen as

$$F_0(k) = U\mathrm{e}^{-\mathrm{i}\kappa(x-2\mu t)\sigma_3} \tag{11}$$

where

$$U = \tfrac{1}{2}\{I - \mathrm{i}(\sigma_1+\sigma_2+\sigma_3)\}. \tag{12}$$

3. Darboux transformation matrix

We define the Jost solutions $F_n(k)$ by the Darboux matrices $D_n(k)$ in a recursive manner:

$$F_n(k) = D_n(k)F_{n-1}(k) \qquad n = 1,2,\ldots \tag{13}$$

where $D_n(k)$ has poles, as we shall discuss later. The properties of $D_n(k)$ and its relation to the solution S of (1) will be determined.

It is obvious that

$$\mu(-\bar{k}) = \overline{\mu(k)} \qquad \kappa(-\bar{k}) = -\overline{\kappa(k)} \tag{14}$$

and then

$$L(-\bar{k}) = \sigma_1\overline{L(k)}\sigma_1 \qquad M(-\bar{k}) = \sigma_1\overline{M(k)}\sigma_1. \tag{15}$$

From (11) we can see that

$$F_0(-\bar{k}) = -\mathrm{i}\sigma_1\overline{F_0(k)}. \tag{16}$$

Hence we have

$$F_n(-\bar{k}) = -\mathrm{i}\sigma_1\overline{F_n(k)} \tag{17}$$

$$D_n(-\bar{k}) = \sigma_1\overline{D_n(k)}\sigma_1. \tag{18}$$

Suppose k_n is a simple pole of $D_n(k)$. Then from (18), $-\bar{k}_n$ is also a pole of $D_n(k)$. If $D_n(k)$ has only these two simple poles we have

$$D_n(k) = C_nB_n(k) \tag{19}$$

where

$$B_n(k) = \left\{ I + \frac{k_n - \bar{k}_n}{k - k_n} B_n + \frac{-\bar{k}_n + k_n}{k + \bar{k}_n} \tilde{B}_n \right\} \tag{20}$$

and C_n is a 2×2 matrix independent of k and

$$(k_n - \bar{k}_n) C_n B_n = D_n \qquad (-\bar{k}_n + k_n) C_n \tilde{B}_n = \tilde{D}_n \tag{21}$$

are residues at poles k_n and $-\bar{k}_n$, respectively. From (18) we have

$$C_n = \sigma_1 \overline{C}_n \sigma_1 \qquad \tilde{B}_n = \sigma_1 \overline{B}_n \sigma_1. \tag{22}$$

From (3), (4) and (11) we can see that

$$L(k) = -L^\dagger(\bar{k}) \qquad M(k) = -M^\dagger(\bar{k}) \tag{23}$$

$$F_0^{-1}(k) = F_0^\dagger(\bar{k}). \tag{24}$$

Hence, we have

$$F_n^{-1}(k) = F_n^\dagger(\bar{k}) \tag{25}$$

$$D_n^{-1}(k) = D_n^\dagger(\bar{k}). \tag{26}$$

From (19) we obtain

$$D_n^{-1}(k) = B_n^{-1}(k) C_n^{-1}. \tag{27}$$

With (26) we obtain

$$C_n^{-1} = C_n^\dagger \tag{28}$$

and

$$B_n^{-1}(k) = B_n^\dagger(\bar{k}) \tag{29}$$

where

$$B_n^\dagger(\bar{k}) = \left\{ I + \frac{\bar{k}_n - k_n}{k - \bar{k}_n} B_n^\dagger + \frac{-k_n + \bar{k}_n}{k + k_n} \sigma_1 B_n^{\mathrm{T}} \sigma_1 \right\}. \tag{30}$$

Since

$$D_n(k) D_n^{-1}(k) = D_n^{-1}(k) D_n(k) = I \tag{31}$$

it has no poles, i.e.

$$B_n B_n^\dagger(\bar{\bar{k}}_n) = 0 \tag{32}$$

and

$$B_n \left\{ I - B_n^\dagger + \frac{-k_n + \bar{k}_n}{2k_n} \sigma_1 B_n^{\mathrm{T}} \sigma_1 \right\} = 0 \tag{33}$$

where the superscript T means transpose. It shows the degeneracy of B_n. One can write

$$B_n = \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \begin{pmatrix} \gamma_n & \delta_n \end{pmatrix}. \tag{34}$$

Substituting these into (33) we obtain a system of linear equations:

$$\gamma_n - \left(|\gamma_n|^2 + |\delta_n|^2 \right) \bar{\alpha}_n - \frac{k_n - \bar{k}_n}{k_n} \gamma_n \delta_n \beta_n = 0 \tag{35}$$

and

$$\delta_n - \left(|\gamma_n|^2 + |\delta_n|^2 \right) \bar{\beta}_n - \frac{k_n - \bar{k}_n}{k_n} \gamma_n \delta_n \alpha_n = 0. \tag{36}$$

Solving these equations one can express α_n and β_n in terms of γ_n and δ_n. We then find

$$B_n = \frac{k}{\Delta_n}\begin{pmatrix} \bar{k}_n|\gamma_n|^2 + k_n|\delta_n|^2 & 0 \\ 0 & \bar{k}_n|\delta_n|^2 + k_n|\gamma_n|^2 \end{pmatrix}\begin{pmatrix} \bar{\gamma}_n \\ \bar{\delta}_n \end{pmatrix}(\gamma_n \quad \delta_n) \tag{37}$$

and

$$\tilde{B}_n = \frac{\bar{k}}{\Delta_n}\begin{pmatrix} \bar{k}_n|\gamma_n|^2 + k_n|\delta_n|^2 & 0 \\ 0 & \bar{k}_n|\delta_n|^2 + k_n|\gamma_n|^2 \end{pmatrix}\begin{pmatrix} \delta_n \\ \gamma_n \end{pmatrix}(\bar{\delta}_n \quad \bar{\gamma}_n) \tag{38}$$

where

$$\begin{aligned}\Delta_n &= |k_n|^2(|\gamma_n|^2 + |\delta_n|^2)^2 - |k_n - \bar{k}_n|^2|\gamma_n|^2|\delta_n|^2 \\ &= (\bar{k}_n|\gamma_n|^2 + k_n|\delta_n|^2)(\bar{k}_n|\delta_n|^2 + k_n|\gamma_n|^2).\end{aligned} \tag{39}$$

Hence, $B_n(k)$ can be expressed as

$$\begin{aligned}B_n(k) = &\frac{1}{(k-k_n)(k+\bar{k}_n)\Delta_n}\begin{pmatrix} \bar{k}_n|\gamma_n|^2 + k_n|\delta_n|^2 & 0 \\ 0 & \bar{k}_n|\delta_n|^2 + k_n|\gamma_n|^2 \end{pmatrix} \\ &\times \left\{k^2\begin{pmatrix} \bar{k}_n|\delta_n|^2 + k_n|\gamma_n|^2 & 0 \\ 0 & \bar{k}_n|\gamma_n|^2 + k_n|\delta_n|^2 \end{pmatrix} + k(k_n^2 - \bar{k}_n^2)\begin{pmatrix} 0 & \bar{\gamma}_n\delta_n \\ \bar{\delta}_n\gamma_n & 0 \end{pmatrix}\right. \\ &\left. -|k_n|^2\begin{pmatrix} \bar{k}_n|\gamma_n|^2 + k_n|\delta_n|^2 & 0 \\ 0 & \bar{k}_n|\delta_n|^2 + k_n|\gamma_n|^2 \end{pmatrix}\right\}.\end{aligned} \tag{40}$$

To determine γ_n and δ_n, we substitute (13) into (8) and (9) with suitable subscripts and then obtain

$$\partial_x D_n(k) = L_n(k)D_n(k) - D_n(k)L_{n-1}(k) \tag{41}$$

$$\partial_t D_n(k) = M_n(k)D_n(k) - D_n(k)M_{n-1}(k). \tag{42}$$

By substituting (19) into (41), and taking the limit as $k \to k_n$, we obtain

$$\partial_x\{C_nB_n\} = L_n(k_n)C_nB_n - C_nB_nL_{n-1}(k_n) \tag{43}$$

and thus

$$\partial_x\{C_nB_nF_{n-1}(k_n)\} = L_n(k_n)C_nB_nF_{n-1}(k_n). \tag{44}$$

Because of the degeneracy of B_n, the second factor on the right-hand side, i.e.

$$(\gamma_n \quad \delta_n)\,F_{n-1}(k_n) \tag{45}$$

must appear in the left-hand side in its original form and, hence, it is independent of x. Similarly, from (42) we obtain

$$\partial_t\{C_nB_nF_{n-1}(k_n)\} = M_n(k_n)C_nB_nF_{n-1}(k_n) \tag{46}$$

which shows that the factor is independent of t. We simply obtain

$$(\gamma_n \quad \delta_n) = (b_n \quad 1)\,F_{n-1}^{-1}(k_n). \tag{47}$$

Here b_n is a constant determined by the boundary condition (10) and the initial condition which we shall see later. Hence, the Darboux matrices $D_n(k)$ have been determined recursively, except for C_n. By a simple algebraic procedure, Δ_n is always non-vanishing regardless of the values x and t. This shows the regularity of B_n and then $B_n(k)$.

4. Relation to the solutions

In the limit as $k \to 1$, from (6) and (7) we have

$$\mu(k), \kappa(k) \to 2\rho \frac{1}{k-1} + \mathrm{O}(1) \tag{48}$$

and then from (41) we obtain

$$(S_n \cdot \sigma) = D_n(1)(S_{n-1} \cdot \sigma) D_n^\dagger(1) \tag{49}$$

in consideration of (25). From (27) we have

$$C_n C_n^\dagger = I. \tag{50}$$

Equations (30) and (50) yield

$$(C_n)_{12} = (C_n)_{21} = 0 \tag{51}$$

and

$$(C_n)_{11} = \overline{(C_n)}_{22} \qquad |(C_n)_{11}| = 1. \tag{52}$$

Alternatively, in the limit as $k \to -1$, from (6) and (7) we have

$$\mu(k) \to -2\rho \frac{1}{k+1} + \mathrm{O}(1) \qquad \kappa(k) \to 2\rho \frac{1}{k+1} + \mathrm{O}(1) \tag{53}$$

and then from (41) we have

$$\sigma_3(S_n \cdot \sigma)\sigma_3 = D_n(-1)\sigma_3(S_{n-1} \cdot \sigma)\sigma_3 D_n^\dagger(-1). \tag{54}$$

The equivalence of this equation with (50) can be shown, since from (18) we have

$$\sigma_3(S \cdot \sigma)\sigma_3 = -\sigma_1 \overline{(S \cdot \sigma)}\sigma_1. \tag{55}$$

The expression for $B_n(1)$ can be obtained from (40):

$$\begin{aligned} B_n(1) = {} & \frac{1}{(1-k_n)(1+\bar{k}_n)\Delta_n} \begin{pmatrix} \bar{k}_n|\gamma_n|^2 + k_n|\delta_n|^2 & 0 \\ 0 & \bar{k}_n|\delta_n|^2 + k_n|\gamma_n|^2 \end{pmatrix} \\ & \times \begin{pmatrix} (1-k_n^2)\bar{k}_n|\delta_n|^2 + (1-\bar{k}_n^2)k_n|\gamma_n|^2 & (k_n^2 - \bar{k}_n^2)\bar{\gamma}_n\delta_n \\ (k_n^2 - \bar{k}_n^2)\bar{\delta}_n\gamma_n & (1-k_n^2)\bar{k}_n|\gamma_n|^2 + (1-\bar{k}_n^2)k_n|\delta_n|^2 \end{pmatrix}. \end{aligned} \tag{56}$$

5. Properties of C_n

Only $(C_n)_{11}$ which has a modulus equal to 1 has to be determined. We write

$$C_n = \mathrm{e}^{\mathrm{i}\omega_n\sigma_3/2} \tag{57}$$

where ω_n is real and characterizes the rotation-angle of spin in the 12-plane. It is necessary to mention that the ω_n may be dependent on x and t. However, we shall see that the condition

$$\omega_n \to 0 \qquad \text{as } |x| \to \pm\infty \tag{58}$$

ensures that the asymptotic spin is aligned on the 1-axis.

Since only relative values of $(b_n \quad 1)$ have meaning, we can find

$$(\gamma_1 \quad \delta_1) \sim (f_1 \quad f_1^{-1}) \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} \tag{59}$$

where

$$f_1 = b_1^{1/2} e^{i\kappa_1(x-2\mu_1 t)} \tag{60}$$

since the last factor on the right-hand side of (59) is proportional to U^{-1}.

Suppose $\operatorname{Im}\kappa$ is positive, then in the limit as $x \to \infty$, $f_1 \to 0$, and

$$B_1(1) \sim \frac{1}{(1-k_1)(1+\bar{k}_1)} \begin{pmatrix} \bar{k}_1 + k_1 & 0 \\ 0 & \bar{k}_1 + k_1 \end{pmatrix} \times \begin{pmatrix} (1-k_1^2)\bar{k}_1 + (1-\bar{k}_1^2)k_1 & -(k_1^2 - \bar{k}_1^2) \\ -(k_1^2 - \bar{k}_1^2) & (1-k_1^2)\bar{k}_1 + (1-\bar{k}_1^2)k_1 \end{pmatrix}. \tag{61}$$

It is a linear combination of I and σ_1, and then commutes with σ_1. Therefore, in the limit as $x \to \infty$,

$$(S_1 \cdot \sigma) = D_1(1)\sigma_1 D_1^\dagger(1) = \sigma_1. \tag{62}$$

One can show step-by-step that in the limit as $x \to \infty$

$$(S_n \cdot \sigma) = \sigma_1. \tag{63}$$

The same results can be obtained in the limit as $x \to -\infty$. Thus (58) is correct.

6. Complete determination of the Darboux transformation matrix

To determine ω_n, one must examine the Lax equations carefully. Since $e^{i\omega_n\sigma_3/2}$ denotes a rotation around the 3-axis, it does not effect the value of S_3. Substituting (19) into (41), and taking the limits as $k \to \infty$ and $k \to 0$ respectively, we obtain

$$\partial_x\{C_n\} = -i2\rho(S_n)_3\sigma_3\{C_n\} + \{C_n\}i2\rho(S_{n-1})_3\sigma_3 \tag{64}$$

$$\partial_x\{C_n B_n(0)\} = i2\rho(S_n)_3\sigma_3\{C_n B_n(0)\} - \{C_n B_n(0)\}i2\rho(S_{n-1})_3\sigma_3. \tag{65}$$

Comparing these two equations, we derive

$$C_n = B_n(0)^{-1/2}. \tag{66}$$

From (40) we have

$$B_n(0) = \frac{1}{\Delta_n} \begin{pmatrix} (\bar{k}_n|\gamma_n|^2 + k_n|\delta_n|^2)^2 & 0 \\ 0 & (\bar{k}_n|\delta_n|^2 + k_n|\gamma_n|^2)^2 \end{pmatrix}. \tag{67}$$

Hence, we obtain

$$C_n = \Delta_n^{-1/2} \begin{pmatrix} \bar{k}_n|\delta_n|^2 + k_n|\gamma_n|^2 & 0 \\ 0 & \bar{k}_n|\gamma_n|^2 + k_n|\delta_n|^2 \end{pmatrix}. \tag{68}$$

This gives the expression for ω_n:

$$\tfrac{1}{2}\omega_n = \arctan\left\{ \frac{k_n''}{k_n'} \frac{|\gamma_n|^2 - |\delta_n|^2}{|\gamma_n|^2 + |\delta_n|^2} \right\} \tag{69}$$

where the superscripts $'$ and $''$ denote the real and imaginary parts of a constant, respectively.

With these expressions we obtain

$$D_n(1) = \frac{1}{(1-k_n)(1+\bar{k}_n)\Delta_n^{1/2}} \times \begin{pmatrix} (1-k_n^2)\bar{k}_n|\delta_n|^2 + (1-\bar{k}_n^2)k_n|\gamma_n|^2 & (k_n^2 - \bar{k}_n^2)\bar{\gamma}_n\delta_n \\ (k_n^2 - \bar{k}_n^2)\bar{\delta}_n\gamma_n & (1-k_n^2)\bar{k}_n|\gamma_n|^2 + (1-\bar{k}_n^2)k_n|\delta_n|^2 \end{pmatrix}. \tag{70}$$

Hence, we have completely determined the Darboux matrices in a recursive manner.

7. The one-soliton solution

Setting $n = 1$ and substituting (70) into (49) we obtain

$$(S_1)_3 = D_1(1)_{12}\overline{D_1(1)}_{11} + D_1(1)_{11}\overline{D_1(1)}_{12} \tag{71}$$

$$(S_1)_1 - \mathrm{i}(S_1)_2 = D_1(1)_{12}\overline{D_1(1)}_{21} + D_1(1)_{11}\overline{D_1(1)}_{22}. \tag{72}$$

In the case of $n = 1$, (59) gives

$$\gamma_1 = f_1 + \mathrm{i}f_1^{-1} \qquad \delta_1 = f_1 - \mathrm{i}f_1^{-1} \tag{73}$$

where

$$f_1^2 = \mathrm{e}^{-\Theta_1}\mathrm{e}^{\mathrm{i}\Phi_1} \tag{74}$$

$$\Phi_1 = 2\kappa_1' x - 2(\kappa_1'\mu_1' - \kappa_1''\mu_1'')t + \Phi_{10} \tag{75}$$

$$\Theta_1 = 2\kappa_1''(x - V_1 t - x_1) \tag{76}$$

$$V_1 = \mu_1' + \frac{\kappa_1'}{\kappa_1''}\mu_1''. \tag{77}$$

Substituting these formulae into (71) and (72) we obtain

$$(S_1)_1 = 1 - 2\frac{\frac{4k_1''^2}{|1-k_1^2|^2} + \frac{k_1''^2}{k_1'^2}\sin^2\Phi_1}{\cosh^2\Theta_1 + \frac{k_1''^2}{k_1'^2}\sin^2\Phi_1} \tag{78}$$

$$(S_1)_2 = \frac{2\frac{4k_1''^2}{|1-k_1^2|^2}\sinh\Theta_1\cos\Phi_1 - 2\frac{k_1''}{k_1'}\frac{1-|k_1|^4}{|1-k_1^2|^2}\cosh\Theta_1\sin\Phi_1}{\cosh^2\Theta_1 + \frac{k_1''^2}{k_1'^2}\sin^2\Phi_1} \tag{79}$$

$$(S_1)_3 = \frac{2\frac{2k_1''(1-|k_1|^2)}{|1-k_1^2|^2}\cosh\Theta_1\cos\Phi_1 + 2\frac{2k_1''^2(1+|k_1|^2)}{k_1'|1-k_1^2|^2}\sinh\Theta_1\sin\Phi_1}{\cosh^2\Theta_1 + \frac{k_1''^2}{k_1'^2}\sin^2\Phi_1}. \tag{80}$$

These are the expressions of the one-soliton solution for a spin chain with an easy plane which have never previously been found by any means. They cannot be obviously factorized into forms of separated variables even in moving coordinates. Hence, it is pointless to solve the Landau–Lifschitz equation for a spin chain with an easy plane by means of separating variables.

In polar coordinates, with the 1-axis being the polar axis, namely $(S_1)_1 = \cos\theta$, from (78) we have

$$\tan^2\frac{\theta}{2} = \left(\frac{4k_1''^2}{|1-k_1^2|^2} + \frac{k_1''^2}{k_1'^2}\sin^2\Phi_1\right) \Big/ \left(\cosh^2\Theta_1 - \frac{4k_1''^2}{|1-k_1^2|^2}\right). \tag{81}$$

8. A system of linear equations

We have given a recursive determination of the Jost solutions. However, one may try to construct a direct procedure for giving multi-soliton solutions. From (13) we define

$$F_N(k) = G_N(k)F_0(k) \tag{82}$$

$$G_N(k) = D_N(k)D_{N-1}(k)\cdots D_1(k). \tag{83}$$

Since $G_N(k)$ has N pairs of poles, k_n and $-\bar{k}_n$, $n = 1, 2, \ldots, N$, we write

$$G_N(k) = K_N A_N(k) \tag{84}$$

where

$$A_N(k) = I + \sum_{n=1}^{N} \frac{1}{k - k_n} A_n + \sum_{n=1}^{N} \frac{1}{k + \bar{k}_n} \tilde{A}_n \tag{85}$$

and K_N is a 2×2 matrix independent of k:

$$K_N = C_N C_{N-1} \cdots C_1 \tag{86}$$

i.e.

$$K_N = \mathrm{e}^{\mathrm{i}\Omega_N \sigma_3/2} \tag{87}$$

$$\Omega_N = \sum_{n=1}^{N} \omega_n. \tag{88}$$

A_n and $\tilde{A}_n$ can be expressed in terms of the above well-defined quantities. For example, we have

$$\begin{aligned} A_n &= K_N^{-1} \lim_{k \to k_n} (k - k_n) D_N(k) D_{N-1}(k) \cdots D_1(k) \\ &= K_N^{-1} D_N(k_n) \cdots D_{n+1}(k_n) C_n (k_n - \bar{k}_n) B_n D_{n-1}(k_n) \cdots D_1(k_n). \end{aligned} \tag{89}$$

However, some properties one can derive directly. From (16) and (17 we have

$$G_N(k) = \sigma_1 \overline{G_N(-\bar{k})} \sigma_1 \tag{90}$$

and then

$$\tilde{A}_n = -\sigma_1 \bar{A}_n \sigma_1. \tag{91}$$

As in (26) and (27), we have

$$G_N^{-1}(k) = G_N^{\dagger}(\bar{k}) \tag{92}$$

and then

$$A_N^{-1}(k) = A_N^{\dagger}(\bar{k}) \tag{93}$$

where

$$A_N^{\dagger}(\bar{k}) = I + \sum_{n=1}^{N} \frac{1}{k - \bar{k}_n} A_n^{\dagger} - \sum_{n=1}^{N} \frac{1}{k + k_n} \sigma_1 A_n^{\mathrm{T}} \sigma_1. \tag{94}$$

From

$$G_N(k) G_N^{-1}(k) = G_N^{-1}(k) G_N(k) = I \tag{95}$$

the residue at $k = k_n$ must vanish:

$$A_m A_N^{\dagger}(\bar{k}_m) = 0 \tag{96}$$

i.e.

$$A_m \left(I + \sum_{n=1}^{N} \frac{1}{k_m - \bar{k}_n} A_n^{\dagger} - \sum_{n=1}^{N} \frac{1}{k_m + k_n} \sigma_1 A_n^{\mathrm{T}} \sigma_1 \right) = 0. \tag{97}$$

This shows the degeneracy of A_m, and one can write

$$A_n = \begin{pmatrix} \alpha'_n \\ \beta'_n \end{pmatrix} (\gamma'_n \quad \delta'_n). \tag{98}$$

Alternatively, we have

$$\partial_x F_N(k) = L_N(k) F_N(k). \tag{99}$$

Hence, we have

$$G_{Nx}(k) - G_N(k)L_0(k) = L_N(k)G_N(k). \tag{100}$$

In the limit as $k \to 1$ we obtain

$$(S_N \cdot \sigma) = G_N(1)\sigma_1 G_N^\dagger(1). \tag{101}$$

Similarly, in the limit as $k \to -1$, we have

$$-\sigma_3(S_N \cdot \sigma)\sigma_3 = G_N(-1)\sigma_1 G_N^\dagger(-1) \tag{102}$$

which is equivalent to (101) on account of (54) and (90).

In the limit as $k \to k_n$, from (99) we have

$$\partial_x \{K_N A_n F_0(k_n)\} = L_N(k_n)\{K_N A_n F_0(k_n)\}. \tag{103}$$

Since A_n is degenerate, the factor

$$(\gamma_n' \quad \delta_n')\, F_0(k_n) \tag{104}$$

must be independent of x. From

$$\partial_t F_N(k) = M_N(k)F_N(k) \tag{105}$$

a similar procedure yields that the factor (104) is also independent of t. Hence, we find

$$(\gamma_n' \quad \delta_n') = (b_n \quad 1)\, F_0^{-1}(k_n) \tag{106}$$

where b_n is a constant which can be shown to appear in (46). We note that α_n', β_n', γ_n' and δ_n' are different from α_n, β_n, γ_n and δ_n, except that

$$\gamma_1' = \gamma_1 \qquad \delta_1' = \delta_1. \tag{107}$$

Substituting (98) into (97) we obtain

$$\gamma_m' = -\sum_{n=1}^{N} \frac{1}{k_m - \bar{k}_n}\left(\gamma_m'\bar{\gamma}_n' + \delta_m'\bar{\delta}_n'\right)\bar{\alpha}_n' + \sum_{n=1}^{N} \frac{1}{k_m + k_n}\left(\gamma_m'\delta_n' + \delta_m'\gamma_n'\right)\beta_n' \tag{108}$$

$$\delta_m' = -\sum_{n=1}^{N} \frac{1}{k_m - \bar{k}_n}\left(\gamma_m'\bar{\gamma}_n' + \delta_m'\bar{\delta}_n'\right)\bar{\beta}_n' + \sum_{n=1}^{N} \frac{1}{k_m + k_n}\left(\gamma_m'\delta_n' + \delta_m'\gamma_n'\right)\alpha_n'. \tag{109}$$

Hence, one can determine the expression for $A_N(k)$.

We now determine K_N. From (99), in the limit as $k \to \infty$, we obtain

$$\partial_x\{K_N\} = -\mathrm{i}2\rho(S_N)_3\sigma_3\{K_N\}. \tag{110}$$

In the limit as $k \to 0$, we have

$$\partial_x\{K_N A_N(0)\} = \mathrm{i}2\rho(S_N)_3\sigma_3\{K_N A_N(0)\}. \tag{111}$$

Comparing these two equations leads to

$$K_N = A_N(0)^{-1/2}. \tag{112}$$

From (87), the meaning of Ω_N is an additional rotation angle around the 3-axis which does not affect the value of S_3.

9. Demonstration

Now we must show that the Jost solution $F_N(k)$ satisfies the corresponding Lax equations. Consider $F_{Nx}(k)F_N^{-1}(k)$. Since $F_N(k)$ and $F_N^{-1}(k)$ have poles k_n, $-\bar{k}_n$, $\bar{k}_n$, $-k_n$, $n = 1, 2, \ldots, N$, one can show that $F_{Nx}(k)F_N^{-1}(k)$ is analytic at these points, i.e. the residues at these points vanish. For example, at point k_n we have

$$\lim_{k\to k_n} \partial_x F_N(k) = \lim_{k\to k_n} \frac{1}{k-k_n}\partial_x \left\{D_N(k_n)\cdots D_{n+1}(k_n)C_n B_n F_{n-1}(k_n)\right\}. \quad (113)$$

Since B_n is degenerate, as we have shown, the factor in (50) is independent of x, hence (113) is equal to

$$\lim_{k\to k_n} \frac{1}{k-k_n}\partial_x \left\{D_N(k_n)\cdots D_{n+1}(k_n)C_n \begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix}\right\}\{(\gamma_n \quad \delta_n)\, F_{n-1}(k_n)\}. \quad (114)$$

Multiplying $F_N^{-1}(k_n)$ from the right, the resulting product vanishes on account of (36). We thus show that the residue of $F_{Nx}(k)F_N^{-1}(k)$ vanishes at point $k = k_n$. Similarly, residues vanish for other points.

We have

$$\partial_x\{F_N(k)\}F_N^{-1}(k) = G_{Nx}G_N^{-1}(k) - \mathrm{i}\mu G_N(k)\sigma_1 G_N^{-1}(k). \quad (115)$$

The right-hand side has poles at $k = 1$ and $k = -1$. At these poles the right-hand side is equal to

$$\begin{cases} -\mathrm{i}2\rho\dfrac{1}{k-1}G_N(1)\sigma_1 G_N^{-1}(1) & \text{when } k\to 1\\ -\mathrm{i}2\rho\dfrac{1}{k+1}G_N(-1)\sigma_1 G_N^{-1}(-1) & \text{when } k\to -1. \end{cases} \quad (116)$$

When $k\to\infty$, the right-hand side of (115) is

$$K_{Nx}K_N^{-1} = \mathrm{i}\tfrac{1}{2}\Omega_{Nx}\sigma_3. \quad (117)$$

Adding these three terms, we obtain

$$L'(k) = -\mathrm{i}4\rho\frac{k}{k^2-1}((S_N)_1\sigma_1 + (S_N)_2\sigma_2) - \mathrm{i}4\rho\frac{1}{k^2-1}(S_N)_3\sigma_3 + \mathrm{i}\tfrac{1}{2}\Omega_{Nx}\sigma_3 \quad (118)$$

where we have used (102). The quantity $F_{Nx}(k)F_N^{-1}(k) - L'(k)$ is analytical in the whole complex k-plane and tends to zero, hence by the Liouville theorem it is equal to zero:

$$\partial_x\{F_N(k)\}F_N^{-1}(k) - L'(k) = 0. \quad (119)$$

When $k\to 0$ this gives

$$\partial_x\{K_N A_N(0)\}\{K_N A_N(0)\}^{-1} - L'(0) = 0. \quad (120)$$

With (118), (120) is

$$-\mathrm{i}\tfrac{1}{2}\Omega_{Nx}\sigma_3 - \{\mathrm{i}4\rho(S_N)_3\sigma_3 + \mathrm{i}\tfrac{1}{2}\Omega_{Nx}\sigma_3\} = 0. \quad (121)$$

From this equation, Ω_{Nx} can be obtained. Substituting the result into $L'(k)$ we find

$$L'(k) = L_N(k). \quad (122)$$

Therefore, we have shown the Jost solution $F_N(k)$ satisfies the first Lax equation.

Similarly, the quantity $F_{Nt}(k)F_N^{-1}(k)$ is analytical at k_n, $-\bar{k}_n$, $\bar{k}_n$, and $-k_n$, $n = 1, 2, \ldots, N$. We have

$$\partial_t\{F_N(k)\}F_N^{-1}(k) = G_{Nt}G_N^{-1}(k) + \mathrm{i}2\mu\kappa G_N(k)\sigma_1 G_N^{-1}(k). \quad (123)$$

In the neighbourhood of $k = 1$ we have

$$\begin{aligned}
&\mathrm{i}2\mu\kappa G_N(k)\sigma_1 G_N^{-1}(k) \\
&\quad \sim \mathrm{i}8\rho^2 \frac{k}{(k-1)^2} G_N(1)\sigma_1 G_N^{-1}(1) + \mathrm{i}8\rho^2 \frac{1}{k-1}\frac{\mathrm{d}}{\mathrm{d}k}\{G_N(k)\sigma_1 G_N^{-1}(k)\}\Big|_{k=1} \\
&\quad + \cdots .
\end{aligned} \tag{124}$$

In the neighbourhood of $k = -1$ we have

$$\begin{aligned}
&\mathrm{i}2\mu\kappa G_N(k)\sigma_1 G_N^{-1}(k) \\
&\quad \sim \mathrm{i}8\rho^2 \frac{k}{(k+1)^2} G_N(-1)\sigma_1 G_N^{-1}(-1) - \mathrm{i}8\rho^2 \frac{1}{k+1}\frac{\mathrm{d}}{\mathrm{d}k}\{G_N(k)\sigma_1 G_N^{-1}(k)\}\Big|_{k=-1} \\
&\quad + \cdots .
\end{aligned} \tag{125}$$

The sum of the first terms of the right-hand sides of these two equations is obviously equal to $-2\kappa L_N(k)$. When $k \to \infty$, the second term on the right-hand side of (121) vanishes and the first term tends to

$$G_{Nt}(k)G_N^{-1}(k) \sim \mathrm{i}\tfrac{1}{2}\Omega_{Nt}\sigma_3. \tag{126}$$

Denote the sum of the five terms on the right-hand side of these three formulae by $M'(k)$. Then $F_{Nt}(k)F_N^{-1}(k) - M'(k)$ is analytical in the whole complex k-plane and tends to zero in the limit as $k \to \infty$. By the Liouville theorem it is equal to zero:

$$\partial_t\{F_N(k)\}F_N^{-1}(k) - M'(k) = 0. \tag{127}$$

We turn to show

$$M'(k) = M_N(k). \tag{128}$$

From the first Lax equation we have

$$G_{Nx}(k)G_N^{-1}(k) - \mathrm{i}\kappa G_N(k)\sigma_1 G_N^{-1}(k) - L_N(k) = 0. \tag{129}$$

In the neighbourhood of $k = 1$ we find

$$G_{Nx}(1)G_N^{-1}(1) - \mathrm{i}2\rho \frac{\mathrm{d}}{\mathrm{d}k}\{G_N(k)\sigma_1 G_N^{-1}(k)\}\Big|_{k=1} = 0. \tag{130}$$

Similarly, in the neighbourhood of $k = -1$ we have

$$G_{Nx}(-1)G_N^{-1}(-1) - \mathrm{i}2\rho \frac{\mathrm{d}}{\mathrm{d}k}\{G_N(k)\sigma_1 G_N^{-1}(k)\}\Big|_{k=-1} = 0. \tag{131}$$

Alternatively, from the first Lax equation we have

$$\partial_x^2 F_N(k) = \left(L_{Nx}(k) + L_N^2(k)\right) F_N(k). \tag{132}$$

That is

$$G_{Nxx}(k) + 2G_{Nx}(k)L_0(k) + G_N(k)L_0^2(k) = \left(L_{Nx}(k) + L_N^2(k)\right) G_N(k). \tag{133}$$

Multiplying this by $\sigma_1 G_N^{-1}(k)$ from the right we obtain

$$\begin{aligned}
&G_{Nxx}(k)\sigma_1 G_N^{-1}(k) - \mathrm{i}2\kappa G_{Nx}(k)G_N^{-1}(k) - \kappa^2 G_N(k)\sigma_1 G_N^{-1}(k) \\
&\quad = \left(L_{Nx}(k) + L_N^2(k)\right) G_N(k)\sigma_1 G_N^{-1}(k).
\end{aligned} \tag{134}$$

Here we notice that

$$L_N^2(k) = -\kappa^2 - 4\rho^2|(S_N)_3|^2. \tag{135}$$

In the neighbourhood of $k = 1$, the terms proportional to $(k-1)^{-1}$ are

$$-\mathrm{i}4\rho G_{Nx}(1)G_N(1) = -\mathrm{i}2\rho(S_{Nx}\cdot\sigma)G_N(1)\sigma_1 G_N^{-1}(1). \tag{136}$$

Similarly, in the neighbourhood of $k=-1$ we have

$$-\,\mathrm{i}4\rho G_{Nx}(-1)G_N(-1)=\mathrm{i}2\rho\sigma_3(S_{Nx}\cdot\sigma)\sigma_3G_N(-1)\sigma_1G_N^{-1}(-1). \tag{137}$$

Hence from (130), (131), (136) and (137) we have

$$\begin{aligned}
\mathrm{i}8\rho^2\frac{1}{k-1}\frac{\mathrm{d}}{\mathrm{d}k}&\{G_N(k)\sigma_1G_N^{-1}(k)\}\Big|_{k=1}-\mathrm{i}8\rho^2\frac{1}{k+1}\frac{\mathrm{d}}{\mathrm{d}k}\{G_N(k)\sigma_1G_N^{-1}(k)\}\Big|_{k=-1}\\
&=4\rho\frac{1}{k-1}G_{Nx}(1)G_N^{-1}(1)-4\rho\frac{1}{k+1}G_{Nx}(-1)G_N^{-1}(-1)\\
&=2\rho\frac{1}{k-1}(S_{Nx}\cdot\sigma)\{G_N(1)\sigma_1G_N^{-1}(1)\}\\
&\quad+2\rho\frac{1}{k+1}\sigma_3(S_{Nx}\cdot\sigma)\sigma_3\{G_N(-1)\sigma_1G_N^{-1}(-1)\}\\
&=2\rho\frac{1}{k-1}(S_{Nx}\cdot\sigma)(S_N\cdot\sigma)-2\rho\frac{1}{k+1}\sigma_3(S_{Nx}\cdot\sigma)(S_N\cdot\sigma)\sigma_3\\
&=2\rho\frac{2k}{k^2-1}\frac{1}{2}\{(S_{Nx}\cdot\sigma)(S_N\cdot\sigma)-\sigma_3(S_{Nx}\cdot\sigma)(S_N\cdot\sigma)\sigma_3\}\\
&\quad+2\rho\frac{2}{k^2-1}\frac{1}{2}\{(S_{Nx}\cdot\sigma)(S_N\cdot\sigma)+\sigma_3(S_{Nx}\cdot\sigma)(S_N\cdot\sigma)\sigma_3\}.
\end{aligned} \tag{138}$$

Hence, we have

$$\begin{aligned}
M'(k)=-2\kappa L_N(k)&+\kappa\tfrac{1}{2}\{(S_{Nx}\cdot\sigma)(S_N\cdot\sigma)-\sigma_3(S_{Nx}\cdot\sigma)(S_N\cdot\sigma)\sigma_3\}\\
&+(\mu-2\rho)\tfrac{1}{2}\{(S_{Nx}\cdot\sigma)(S_N\cdot\sigma)+\sigma_3(S_{Nx}\cdot\sigma)(S_N\cdot\sigma)\sigma_3\}+\mathrm{i}\tfrac{1}{2}\Omega_{Nt}\sigma_3.
\end{aligned} \tag{139}$$

Substituting this into (127) and setting $k=0$, we obtain

$$\mathrm{i}\tfrac{1}{2}\Omega_{Nt}\sigma_3=2\rho\tfrac{1}{2}\{(S_{Nx}\cdot\sigma)(S_N\cdot\sigma)+\sigma_3(S_{Nx}\cdot\sigma)(S_N\cdot\sigma)\sigma_3\}. \tag{140}$$

Substituting this into (140) we finally obtain (128). Therefore, the Jost solution $F_N(k)$ satisfies the second Lax equation.

10. Matrix form of the equations

To solve equations (97) and (98) we introduce a transformation:

$$G'_N(k)=U^{-1}G_N(k)U. \tag{141}$$

Hence, we have

$$A'_n=U^{-1}A_nU. \tag{142}$$

However, we must define

$$\tilde{A}'_n=U^{-1}\tilde{A}_nU=-U^{-1}\sigma_1\bar{A}_n\sigma_1U=-U^{-1}\sigma_1\overline{UA'_nU^{-1}}\sigma_1U=-\overline{A'}_n \tag{143}$$

since

$$U^{-1}\sigma_1\overline{U}=\mathrm{i}. \tag{144}$$

Corresponding to (96) we have

$$A'_mA'^{\dagger}_N(\bar{k}_m)=0 \tag{145}$$

i.e.

$$A'_m\left(I+\sum_{n=1}^{N}\frac{1}{k_m-\bar{k}_n}A'^{\dagger}_n-\sum_{n=1}^{N}\frac{1}{k_m+k_n}A'^{\mathrm{T}}_n\right)=0. \tag{146}$$

Noticing (106), A'_n can be expressed as

$$A'_n=\begin{pmatrix}g_n\\h_n\end{pmatrix}(f_n\quad f_n^{-1}) \tag{147}$$

where

$$f_n=b_n^{1/2}\mathrm{e}^{\mathrm{i}\kappa_n(x-2\mu_n t)}. \tag{148}$$

Substituting (147) into (146) we obtain

$$f_m=-\sum_{n=1}^{N}\frac{1}{k_m-\bar{k}_n}(f_m\bar{f}_n+f_m^{-1}\bar{f}_n^{-1})\bar{g}_n+\sum_{n=1}^{N}\frac{1}{k_m+k_n}(f_mf_n+f_m^{-1}f_n^{-1})g_n \tag{149}$$

$$f_m^{-1}=-\sum_{n=1}^{N}\frac{1}{k_m-\bar{k}_n}(f_m\bar{f}_n+f_m^{-1}\bar{f}_n^{-1})\bar{h}_n+\sum_{n=1}^{N}\frac{1}{k_m+k_n}(f_mf_n+f_m^{-1}f_n^{-1})h_n. \tag{150}$$

From these one can find g_n and h_n, and the $A'_N(k)$. For example

$$A'_N(1)_{11}=1+\sum_{n=1}^{N}\frac{1}{1-k_n}g_nf_n-\frac{1}{1+\bar{k}_n}\bar{g}_n\bar{f}_n. \tag{151}$$

From (149) we have

$$1=-\sum_{n=1}^{N}\frac{1}{k_m-\bar{k}_n}(1+f_m^{-2}\bar{f}_n^{-2})\bar{g}_n\bar{f}_n+\sum_{n=1}^{N}\frac{1}{k_m+k_n}(1+f_m^{-2}f_n^{-2})g_nf_n. \tag{152}$$

and then

$$1=\sum_{n=1}^{N}\frac{1}{\bar{k}_m+\bar{k}_n}(1+\bar{f}_m^{-2}\bar{f}_n^{-2})\bar{g}_n\bar{f}_n-\sum_{n=1}^{N}\frac{1}{\bar{k}_m-k_n}(1+\bar{f}_m^{-2}f_n^{-2})g_nf_n. \tag{153}$$

From (152) one can find g_n and $\bar{g}_n$ without difficulty, and then $A'_N(k)_{11}$ and $A'_N(k)_{12}$. However, owing to the appearance of g_n and $\bar{g}_n$, $n=1,2,\ldots,N$ in every equation of the system (152), it is difficult to obtain explicit expressions for them by the well known Binet–Cauchy formula. However, the asymptotic behaviour of the solutions can be derived explicitly from them.

Introduce

$$\Psi_l=\begin{cases}g_nf_n & \text{if } n=l,\ l\in 1,2,\ldots,N\\ \bar{g}_n\bar{f}_n & \text{if } n=l-N,\ l\in N+1,N+2,\ldots,2N\end{cases} \tag{154}$$

and

$$E_l=1\qquad l\in 1,2,\ldots,2N \tag{155}$$

where we notice that E is a row matrix. Equations (151) and (152) can be written as

$$E=\Psi Q \tag{156}$$

where Q is a $2N \times 2N$ matrix such that

$$Q_{nm} = \frac{1}{k_n + k_m}(1 + f_n^{-2} f_m^{-2}) \tag{157}$$

$$Q_{N+n,m} = \frac{1}{\bar{k}_n - k_m}(1 + \bar{f}_n^{-2} f_m^{-2}) \tag{158}$$

$$Q_{n,N+m} = \frac{1}{k_n - \bar{k}_m}(1 + f_n^{-2} \bar{f}_m^{-2}) \tag{159}$$

$$Q_{N+n,N+m} = \frac{1}{\bar{k}_n + \bar{k}_m}(1 + \bar{f}_n \bar{f}_m). \tag{160}$$

From (155) we obtain

$$\Psi = EQ^{-l}. \tag{161}$$

$A'_N(1)_{11}$ can be expressed as

$$A'_N(1)_{11} = 1 + \sum_{l=1}^{2N} \Psi_l P_l = 1 + \Psi P^{\mathrm{T}} \tag{162}$$

where

$$P_l = \begin{cases} \dfrac{1}{1 - k_n} & \text{if } n = l,\ l \in \{1, 2, \ldots, N\} \\ -\dfrac{1}{1 + \bar{k}_n} & \text{if } n = l - N,\ l \in \{N + 1, N = 2, \ldots, 2N\}. \end{cases} \tag{163}$$

From (160) and (161) we obtain

$$A'_N(1)_{11} = 1 + EQ^{-1}P^{\mathrm{T}} = 1 + \mathrm{Tr}\{Q^{-1}P^{\mathrm{T}}E\} = \frac{\det(Q + P^{\mathrm{T}}E)}{\det Q} = \frac{\det Q'}{\det Q} \tag{164}$$

where

$$Q' = Q + P^{\mathrm{T}}E. \tag{165}$$

Consider $\det Q$, when $N = 1$, $k = k_j$. Then

$$\begin{aligned} \det Q &= \det \begin{pmatrix} \frac{1}{2k_j}(1 + f_j^{-4}) & \frac{1}{k_j - \bar{k}_j}(1 + |f_j|^{-4}) \\ \frac{1}{\bar{k}_j - k_j}(1 + |f_j|^{-4}) & \frac{1}{2\bar{k}_j}(1 + \bar{f}_j^{-4}) \end{pmatrix} \\ &= -\frac{(k_j + \bar{k}_j)^2}{4|k_j|^2 |k_j - \bar{k}_j|^2}\left(1 + |f_j|^{-8}\right) - \frac{2}{|k_j - \bar{k}_j|^2}|f_j|^{-4} + \frac{1}{4|k_j|^2}\left(f_j^{-4} + \bar{f}_j^{-4}\right). \end{aligned} \tag{166}$$

11. Asymptotic behaviour

From (148) we write

$$f_n^2 = \mathrm{e}^{-\Theta_n}\mathrm{e}^{\mathrm{i}\Phi_n} \tag{167}$$

where

$$\Phi_n = 2\kappa'_n x - 2(\kappa'_n \mu'_n - \kappa''_n \mu'' - n)t + \Phi_{n0} \tag{168}$$

$$\Theta_n = 2\kappa''_n (x - V_n t - x_n) \tag{169}$$

$$V_n = \mu'_n + \frac{\kappa'_n}{\kappa''_n}\mu''_n. \tag{170}$$

Suppose all $\kappa_n'' > 0$, and

$$V_N > V_{N-1} > \cdots > V_1. \tag{171}$$

The vicinity of $V_n t + x_n$ is denoted by Ω_n. In the extreme large t, these vicinities are separated from left to right as

$$\Omega_N, \Omega_{N-1}, \ldots, \Omega_1. \tag{172}$$

In the vicinity Ω_j, we have

$$\begin{cases} x - V_n t - x_n \to -\infty & |f_n|^{-1} \to 0 \qquad \text{if } n < j \\ x - V_m t - x_m \to \infty & |f_m|^{-1} \to \infty \qquad \text{if } m > j. \end{cases} \tag{173}$$

In this limit, $\det Q$ tends to

$$\begin{vmatrix} \frac{1}{k_n+k_{n'}} & \frac{1}{k_n+k_j} & 0 & \frac{1}{k_n-\bar{k}_{n'}} & \frac{1}{k_n-\bar{k}_j} & 0 \\ \frac{1}{k_j+k_{n'}} & \frac{1+f_j^{-4}}{2k_j} & \frac{f_j^{-2} f_{m'}^{-2}}{k_j+k_{m'}} & \frac{1}{k_j-\bar{k}_{n'}} & \frac{1+|f_j|^{-4}}{k_j-\bar{k}_j} & \frac{f_j^{-2}\bar{f}_{m'}^{-2}}{k_j-\bar{k}_{m'}} \\ 0 & \frac{f_m^{-2} f_j^{-2}}{k_m+k_j} & \frac{f_m^{-2} f_{m'}^{-2}}{k_m+k_{m'}} & 0 & \frac{f_m^{-2}\bar{f}_j^{-2}}{k_m-\bar{k}_j} & \frac{f_m^{-2}\bar{f}_{m'}^{-2}}{k_m-\bar{k}_{m'}} \\ \frac{1}{\bar{k}_n-k_{n'}} & \frac{1}{\bar{k}_n-k_j} & 0 & \frac{1}{\bar{k}_n+\bar{k}_{n'}} & \frac{1}{\bar{k}_n+\bar{k}_j} & 0 \\ \frac{1}{\bar{k}_j-k_{n'}} & \frac{1+|f_j|^{-4}}{\bar{k}_j-k_j} & \frac{\bar{f}_j^{-2} f_{m'}^{-2}}{\bar{k}_j-k_{m'}} & \frac{1}{\bar{k}_j+\bar{k}_{n'}} & \frac{1+\bar{f}_j^{-4}}{2\bar{k}_j} & \frac{\bar{f}_j^{-2}\bar{f}_{m'}^{-2}}{\bar{k}_j+\bar{k}_{m'}} \\ 0 & \frac{\bar{f}_m^{-2} f_j^{-2}}{\bar{k}_m-k_j} & \frac{\bar{f}_m^{-2} f_{m'}^{-2}}{\bar{k}_m-k_{m'}} & 0 & \frac{\bar{f}_m^{-2}\bar{f}_j^{-2}}{\bar{k}_m+\bar{k}_j} & \frac{\bar{f}_m^{-2}\bar{f}_{m'}^{-2}}{\bar{k}_m+\bar{k}_{m'}} \end{vmatrix} \tag{174}$$

where

$$n, n' < j \qquad m, m' > j. \tag{175}$$

Here we notice that we only those terms leading to $|f_{j+1}|^{-8} \cdots |f_N|^{-8}$ remain. It is difficult to calculate this determinant. With a procedure similar to that given in the paper by Huang and Chen (1990), consider the term without f_j which is given by

$$\begin{vmatrix} \frac{1}{k_n+k_{n'}} & \frac{1}{k_n+k_j} & 0 & \frac{1}{k_n-\bar{k}_{n'}} & \frac{1}{k_n-\bar{k}_j} & 0 \\ \frac{1}{k_j+k_{n'}} & \frac{1}{2k_j} & 0 & \frac{1}{k_j-\bar{k}_{n'}} & \frac{1}{k_j-\bar{k}_j} & 0 \\ 0 & 0 & \frac{f_m^{-2} f_{m'}^{-2}}{k_m+k_{m'}} & 0 & 0 & \frac{f_m^{-2}\bar{f}_{m'}^{-2}}{k_m-\bar{k}_{m'}} \\ \frac{1}{\bar{k}_n-k_{n'}} & \frac{1}{\bar{k}_n-k_j} & 0 & \frac{1}{\bar{k}_n+\bar{k}_{n'}} & \frac{1}{\bar{k}_n+\bar{k}_j} & 0 \\ \frac{1}{\bar{k}_j-k_{n'}} & \frac{1}{\bar{k}_j-k_j} & 0 & \frac{1}{\bar{k}_j+\bar{k}_{n'}} & \frac{1}{2\bar{k}_j} & 0 \\ 0 & 0 & \frac{\bar{f}_m^{-2} f_{m'}^{-2}}{\bar{k}_m-k_{m'}} & 0 & 0 & \frac{\bar{f}_m^{-2}\bar{f}_{m'}^{-2}}{\bar{k}_m+\bar{k}_{m'}} \end{vmatrix}. \tag{176}$$

The term involving f_j^{-4} is the determinant

$$\begin{vmatrix} \frac{1}{k_n+k_{n'}} & 0 & 0 & \frac{1}{k_n-\bar{k}_{n'}} & \frac{1}{k_n-\bar{k}_j} & 0 \\ 0 & \frac{f_j^{-4}}{2k_j} & \frac{f_j^{-2} f_{m'}^{-2}}{k_j+k_{m'}} & 0 & 0 & \frac{f_j^{-2}\bar{f}_{m'}^{-2}}{k_j-\bar{k}_{m'}} \\ 0 & \frac{f_m^{-2} f_j^{-2}}{k_m+k_j} & \frac{f_m^{-2} f_{m'}^{-2}}{k_m+k_{m'}} & 0 & 0 & \frac{f_m^{-2}\bar{f}_{m'}^{-2}}{k_m-\bar{k}_{m'}} \\ \frac{1}{\bar{k}_n-k_{n'}} & 0 & 0 & \frac{1}{\bar{k}_n+\bar{k}_{n'}} & \frac{1}{\bar{k}_n+\bar{k}_j} & 0 \\ \frac{1}{\bar{k}_j-k_{n'}} & 0 & 0 & \frac{1}{\bar{k}_j+\bar{k}_{n'}} & \frac{1}{2\bar{k}_j} & 0 \\ 0 & \frac{\bar{f}_m^{-2} f_j^{-2}}{\bar{k}_m-k_j} & \frac{\bar{f}_m^{-2} f_{m'}^{-2}}{\bar{k}_m-k_{m'}} & 0 & 0 & \frac{\bar{f}_m^{-2}\bar{f}_{m'}^{-2}}{\bar{k}_m+\bar{k}_{m'}} \end{vmatrix}. \tag{177}$$

In addition to the common factor $|f_{j+1}|^{-8}\cdots|f_N|^{-8}$, these two determinants are clearly proportional to

$$\begin{vmatrix} \frac{1}{k_n+k_{n'}} & \frac{1}{k_n-\bar{k}_{n'}} & \frac{1}{k_n-\bar{k}_j} \\ \frac{1}{\bar{k}_n-k_{n'}} & \frac{1}{\bar{k}_n+\bar{k}_{n'}} & \frac{1}{\bar{k}_n+k_j} \\ \frac{1}{\bar{k}_n-k_{n'}} & \frac{1}{\bar{k}_n+\bar{k}_{n'}} & \frac{1}{2\bar{k}_j} \end{vmatrix} \begin{vmatrix} \frac{1}{k_m+k_{m'}} & \frac{1}{k_m-\bar{k}_{m'}} \\ \frac{1}{\bar{k}_m-k_{m'}} & \frac{1}{\bar{k}_m+\bar{k}_{m'}} \end{vmatrix} \tag{178}$$

where the proportional coefficients are

$$\frac{1}{2k_j}\frac{(k_j+\bar{k}_j)^2}{|k_j-\bar{k}_j|^2}\prod_{n=1}^{j-1}\frac{(k_j-k_n)^2(k_j+\bar{k}_n)^2}{(k_j+k_n)^2(k_j-\bar{k}_n)^2} \tag{179}$$

and

$$-\frac{1}{2k_j}\prod_{m=j+1}^{N}\frac{(k_j-k_m)^2(k_j+\bar{k}_m)^2}{(k_j+k_m)^2(k_j-\bar{k}_m)^2} \tag{180}$$

respectively. Therefore, in this limit the asymptotic behaviour is similar to the one-soliton solution but f_j is replaced by $f_j^{(+)}$ such that

$$\left(f_j^{(+)}\right)^{-2} = f_j^{-2}\alpha_j^{-1}\beta_j \tag{181}$$

where

$$\alpha_j = \prod_{n=1}^{j-1}\frac{(k_j-k_n)(k_j+\bar{k}_n)}{(k_j+k_n)(k_j-\bar{k}_n)} \tag{182}$$

$$\beta_j = \prod_{m=j+1}^{N}\frac{(k_j-k_m)(k_j+\bar{k}_m)}{(k_j+k_m)(k_j-\bar{k}_m)}. \tag{183}$$

Hence, in this limit we have $\det Q \to \det Q_j^{(+)}$, where

$$\det Q_j^{(+)} = -\frac{(k_j+\bar{k}_j)^2}{4|k_j|^2|k_j-\bar{k}_j|^2}\left(1+|f_j^{(+)}|^{-8}\right)+\frac{1}{4|k_j|^2}\left((f_j^{(+)})^{-4}+(\overline{f_j^{(+)}})^{-4}\right). \tag{184}$$

Similarly, in this limit one can obtain the asymptotic expression of $\det Q'$.

The corresponding $\Theta_j^{(+)}$ and $\Phi_j^{(+)}$ differ from those given in (175) and (176) in that

$$\Phi_j^{(+)} = 2\kappa_j' x - 2(\kappa_j'\mu_j' - \kappa_j''\mu_j'')t + \Phi_{j0} + \Gamma_j^{(+)} \tag{185}$$

$$\Theta_j^{(+)} = 2\kappa_j''\left(x - V_j t - x_j - X_j^{(+)}\right) \tag{186}$$

where

$$\Gamma_j^{(+)} = \arg\alpha_j - \arg\beta_j \tag{187}$$

$$X_j^{(+)} = \frac{1}{2\kappa_j}\left(\log|\alpha_j| - \log|\beta_j|\right). \tag{188}$$

Similarly, when $t \to -\infty$ in the vicinity Ω_j, the corresponding behaviour can be obtained. For example, in an analogy to (187) and (188) we have

$$\Gamma_j^{(-)} = -\Gamma_j^{(+)} \tag{189}$$

$$X_j^{(-)} = -X_j^{(+)}. \tag{190}$$

Therefore, the total additional displacement of centre X_j and the total phase shift Γ_j are

$$\Gamma_j = 2\Gamma_j^{(+)} \qquad X_j = 2X_j^{(+)}. \tag{191}$$

12. Relations to the isotropic spin chain

In the limit as $\rho \to 0$, that is, when the anisotropy vanishes, it is convenient to introduce an auxiliary parameter ζ such that

$$\mu = \zeta + \rho^2 \zeta^{-1} \tag{192}$$

$$\kappa = \zeta - \rho^2 \zeta^{-1}. \tag{193}$$

It is obvious that (5) is satisfied. Comparing with (6) and (7) we have

$$k = \frac{\zeta + \rho}{\zeta - \rho}. \tag{194}$$

It is obvious that $\zeta = \pm\rho$ correspond to zero κ and to $\mu = \pm 2\rho$. In the complex μ plane, these two points are the edges of cuts.

Alternatively, $\zeta = \pm\rho$ correspond to $k = \infty$ and $k = 0$. In the above discussion we have seen that they give a contribution which yields the factor C_n in (19) or the factor K_N in (84). These factors are important in order to ensure that the generated Jost solutions satisfy the corresponding Lax equations. This indicates that in the inverse transform the edges of cuts must give a contribution, even in the reflectionless case. Unfortunately, Bolovik and Kulinich (1984) never made any proposals on how to consider these effects. They did not obtain an expression for the solution.

One can then obtain the expression of the one-soliton solution in terms of the parameter ζ. We restrict ζ_1 in the upper-half plane of complex ζ, and

$$|\zeta_1| > \rho. \tag{195}$$

Then, from (194) we find

$$k_1'' = 2\rho \frac{\zeta_1''}{|\zeta_1 - \rho^2|^2} \tag{196}$$

and

$$k_1' = \epsilon \frac{|\zeta_1|^2 - \rho^2}{|\zeta_1 - \rho|^2} \tag{197}$$

where $\epsilon = \pm 1$ correspond to $k_1' > 0$ and $k_1' < 0$, respectively. We obtain

$$(S_1)_1 = 1 - 2\frac{\frac{\zeta_1''^2}{|\zeta_1|^2} + \frac{4\rho^2\zeta_1''^2}{(|\zeta_1|^2-\rho^2)^2}\sin^2\Phi_1}{\cosh^2\Theta_1 + \frac{4\rho^2\zeta_1''^2}{(|\zeta_1|^2-\rho^2)^2}\sin^2\Phi_1} \tag{198}$$

$$(S_1)_2 = \frac{2\frac{\zeta_1''^2}{|\zeta_1|^2}\sinh\Theta_1\cos\Phi_1 - 2\frac{\zeta_1'\zeta_1''(|\zeta_1|^2+\rho^2)}{|\zeta_1|^2(|\zeta_1|^2-\rho^2)}\cosh\Theta_1\sin\Phi_1}{\cosh^2\Theta_1 + \frac{4\rho^2\zeta_1''^2}{(|\zeta_1|^2-\rho^2)^2}\sin^2\Phi_1} \tag{199}$$

$$(S_1)_3 = \frac{2\frac{\zeta_1'\zeta_1''}{|\zeta_1|^2}\cosh\Theta_1\cos\Phi_1 + 2\frac{\zeta_1''^2(|\zeta_1|^2+\rho^2)}{|\zeta_1|^2(|\zeta_1|^2-\rho^2)}\sinh\Theta_1\sin\Phi_1}{\cosh^2\Theta_1 + \frac{4\rho^2\zeta_1''^2}{(|\zeta_1|^2-\rho^2)^2}\sin^2\Phi_1}. \tag{200}$$

In polar coordinates, with the 1-axis being the polar axis, namely $(S_1)_1 = \cos\theta$, from (198) we have

$$\tan^2\frac{\theta}{2} = \left(\frac{\zeta_1''^2}{|\zeta_1|^2} + \frac{4\rho^2\zeta_1''^2}{(|\zeta_1|^2-\rho^2)^2}\sin^2\Phi_1\right) \Bigg/ \left(\cosh^2\Theta_1 - \frac{\zeta_1''^2}{|\zeta_1|^2}\right). \tag{201}$$

From the expression of $\cos\theta$, (198), we obtain properties of the soliton: (a) the depth of the valley is not a constant but varies with time periodically, (b) the width of the valley is also not a constant but varies with time periodically, (c) the integral

$$\int_{-\infty}^{\infty} \mathrm{d}x\,(1-\cos\theta) \tag{202}$$

is also not a constant and is dependent on time periodically. These are important properties which have never been seen in the other soliton solutions obtained until now.

One can see that when $\rho \to 0$, the expressions (198)–(201) reduce obviously to those for an isotropic chain.

13. Concluding remark

In our view, for the Landau–Lifschitz equation for a continuous spin chain with an easy plane, exact soliton solutions have never previously been found by any means tried. The expressions for soliton solutions obtained in this work may be useful for further theoretical study and practical application.

From the expression of the one-soliton solution (198)–(201), it is obvious that it cannot be expressed in the form of products of separated variables in moving coordinates. Hence, it cannot be found by separating variables.

When the anisotropy vanishes, the solutions obtained tend obviously to those for the isotropic chain. From (201) one can see that the solution given by Kosevich *et al* (1977) does not satisfy the Landau–Lifschitz equation for a spin chain with an easy plane even in the first order of anisotropy, and there is no reason to consider it as an approximate solution.

From the explicit expression for the one-soliton solution, one can see that it is difficult to express it in a Hirota form of factorization. It is not surprising that the work of Bogdan and Kovalev (1980) did not obtain the desired results.

In the works of Bolovik and Kulinich (1984), they developed an inverse transform method. In addition to questions about their derivation, they did not consider the contributions due to the edges of cuts in the complex plane which appear even in the reflectionless case. Hence, they did not obtain any useful final expressions.

It is well known that from the gauge equivalence of the isotropic spin chain to the nonlinear Schrödinger equation with vanishing boundary conditions (Zakharov and Takhtajan 1979), one can obtain the soliton solutions of the former from the soliton solutions of the latter by a gauge transformation. In the work of Nakamura and Sasada (1982), they showed gauge equivalence of the spin chain with an easy plane to the nonlinear Schrödinger equation in a positive dispersion regime ($\mathrm{NLS}^{(+)}$ equation) in the case of a non-vanishing boundary value (say, some constant value c). From the expressions for multi-soliton solutions of the Landau–Lifschitz equation for a spin chain with an easy plane, we have shown that the solutions tend to those for the isotropic spin chain. As we known, the $\mathrm{NLS}^{(+)}$ in the case of a non-vanishing boundary value has dark-soliton solutions; when the boundary value c tends to zero the equation has no non-trivial solutions, except 0. Hence, the gauge equivalence between these two equations may not exist.

The Darboux transformation method in soliton theory was developed a while ago (see, for instance, Matveev and Salle 1991), however, its matrix form was first introduced by Levi *et al* (1981) in solving the KdV equation. Many authors have given contributions in this direction (see, Gu and Zhou 1987, Chen *et al* 1988, 1989, Chau *et al* 1991). For these nonlinear equations, their Lax pairs have a common property in that the Lax pairs are independent of particular solutions of the nonlinear equations in the limit as the spectral parameters tend to infinity, or in the limit as those tend to zero. For the Landau–Lifschitz equation for a spin chain with an easy plane, the Lax pair does not have this property. The present work gives a successful example for the method of Darboux transformation matrix. The problem due to the lack of this property has been solved by detailed investigation of the properties of the Darboux matrix and by the Liouville theorem. These may help the inverse transform method for treating the same problem in its own framework.

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